LAW OF THE EXPONENTIAL FUNCTIONAL OF A NEW FAMILY OF ONE-SIDED LÉVY PROCESSES VIA SELF-SIMILAR CONTINUOUS STATE BRANCHING PROCESSES WITH IMMIGRATION AND THE ${}_p\Psi_q$ WRIGHT HYPERGEOMETRIC FUNCTIONS

P. PATIE[(1)]

We first introduce and derive some basic properties of a two-parameters (α,γ) family of one-sided Lévy processes, with $1<\alpha<2$ and $\gamma>-\alpha$. Their Laplace exponents are given in terms of the Pochhammer symbol as follows

$$\psi^{(\gamma)}(\lambda) \quad = \quad c\left((\lambda+\gamma)_\alpha-(\gamma)_\alpha\right), \quad \lambda \geq 0,$$

where c is a positive constant, $(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)}$ stands for the Pochhammer symbol and Γ for the gamma function. These are a generalization of the Brownian motion, since in the limit case $\alpha \to 2$, we end up to Brownian motion with drift $\gamma+\frac{1}{2}$. Then, we proceed by computing the density of the law of the exponential functional associated to some elements of this family (and their dual) and some transformations of these elements. More specifically, we shall consider the Lévy processes which admit the following Laplace exponent, for any $\delta > \frac{\alpha-1}{\alpha}$,

$$\psi^{(0,\delta)}(\lambda) \quad = \quad \psi^{(0)}(\lambda) - \frac{\alpha\delta}{\lambda+\alpha-1}\psi^{(0)}(\lambda), \quad \lambda \geq 0.$$

These densities are expressed in terms of the Wright hypergeometric function ${}_1\Psi_1$. By means of probabilistic arguments, we derive some interesting properties enjoyed by this function. On the way we also characterize explicitly the semi-group of the family of self-similar positive Markov processes associated, via the Lamperti mapping, to the family of Lévy processes with Laplace exponent $\psi^{(0,\delta)}$.

1. Introduction

The exponential functional of Lévy processes plays an important role from both theoretical and applied perspectives. Indeed, it appears in various fields such as diffusion processes in random environments, fragmentation and coalescence processes, the classical moment problems, mathematical finance, astrophysics ... We refer to the paper of Bertoin and Yor [5] for a thorough survey on this topic and a description of cases when the law of such functional is known explicitly. We also refer, in the case of the Brownian motion with drift, to the two survey papers of Matsumoto and Yor [23] and [24] where the law of the exponential functional allows to characterize several interesting stochastic processes and to develop stochastic analysis related to Brownian motions on hyperbolic spaces. Finally,

Key words and phrases. Lévy processes, self-similarity, continuous state branching processes with immigration, Wright hypergeometric functions.

we mention that Bertoin et al. [3] have expressed the law of the exponential functional of a Poisson point process by means of q-calculus and have derived an interesting connection to the indeterminate moment problem associated to a log-normal random variable.

In this paper, we start by introducing through its Laplace exponent and its characteristics triplet, a two (α, γ)-parameters family of spectrally negative Lévy processes, with $1 < \alpha \leq 2$ and $\gamma > -\alpha$. The Laplace exponent has the following form

$$\psi^{(\gamma)}(\lambda) \;=\; c\left((\lambda+\gamma)_\alpha - (\gamma)_\alpha\right), \quad \lambda \geq 0, \tag{1.1}$$

where c is a positive constant, $(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)}$ stands for the Pochhammer symbol and Γ for the Gamma function. It turns out that this family possesses several interesting properties. Indeed, their Lévy measures behave around 0 as the Lévy measure of a stable Lévy process of index α. Moreover, a member of this family has its law at a fixed time which belongs to the domain of attraction of a stable law of index α. They include the family of Brownian motion with positive drifts but also negative since they admit negative exponential moments. After studying some further basic properties of this family, we compute the density of the law of the exponential functional associated to some elements of this family (and their dual) and some transformations of these elements. More specifically, we shall consider the Lévy processes which admit the following Laplace exponent, for any $\delta > \frac{\alpha-1}{\alpha}$,

$$\psi^{(0,\delta)}(\lambda) \;=\; \psi^{(0)}(\lambda) - \frac{\alpha\delta}{\lambda+\alpha-1}\psi^{(0)}(\lambda), \quad \lambda \geq 0. \tag{1.2}$$

The densities of the corresponding exponential functionals are expressed in terms of the Wright hypergeometric function ${}_1\Psi_1$. As a specific instance, we obtain the inverse Gamma law and hence recover Dufresne's result [11] regarding the law of the exponential functional of a Brownian motion with negative drifts. As a limit case, we show that the family encompasses the inverse Linnik law.

The path we follow to derive such a law is as follows. The first stage consists on characterizing the family of Lévy processes associated, via the Lamperti mapping, to the family of self-similar continuous time branching processes with immigration (for short cbip). We mention that this family includes the family of squared Bessel processes and belong to the class of affine term structure models in mathematical finance, see the survey paper of Duffie et al. [10]. At a second stage, by using well-know results on cbip, we derive the spatial Laplace transforms of the semi-groups of this later family. Then, by means of inversion techniques, we compute, in terms of a power series, the density of these semi-groups. In particular, we get an expression of their entrance laws in terms of the Wright hypergeometric function ${}_1\Psi_1$. Finally, we end up our journey by related these entrance laws to the law of the exponential functionals associated to the family of Lévy processes (1.2).

On the way, we get an expression, as a power series, for the density of the semi-group of a family of sspMp. We mention, that beside the family of Bessel processes, such a semi-group is only known explicitly for the so-called saw-tooth processes (a piecewise linear sspMp) which were studied by Carmona et al. [9]. Therein, the authors use the multiplicative kernel associated to a gamma random variable to obtain an intertwining

relationship between the semi-group of the Bessel processes and the family of piecewise linear sspMp.

The remaining part of the paper is organized as follows. In the next Section, we gather some preliminary results. In particular, we introduce a transformation which leaves invariant the class of sspMp without diffusion coefficients. We also provide the detailed computation of an integral which will appear several times in the paper. Section 3 concerns the definition and the study of some basic properties of the new family of one-sided Lévy processes (1.1). Section 4 is devoted to the statement and the proof of the law of the exponential functionals under study. Section 5 contains several remarks regarding some representations of the special functions which appear in this paper. Finally Section 6 is a summarize of interesting properties enjoy by the Wright hypergeometric functions.

2. **Recalls and preliminary results**

2.1. **Self-similar positive Markov processes.** Let $\zeta = (\zeta_t)_{t\geq 0}$ denotes a real-valued Lévy process starting from x. Then, for any $\alpha > 0$, introduce the time change process

$$X_t \quad = \quad e^{\zeta_{A_t}}$$

where

$$A_t \quad = \quad \inf\{s \geq 0; \Sigma_s := \int_0^s e^{\alpha\zeta_u}\, du > t\}.$$

Lamperti [20] showed that the process $(X_t)_{t\geq 0}$ is a positive-valued $\frac{1}{\alpha}$ semi-stable Markov process (for short sspMp) starting from e^x. That is, if, for $x > 0$, $\mathbb{Q}_x$ denotes the law of X starting from x, then X is a positive valued Markov process which enjoys the α-self-similarity property, i.e. for any $c > 0$, we have the equality in distribution

$$((X_{c^\alpha t})_{t\geq 0}, \mathbb{Q}_{cx}) \stackrel{(d)}{=} ((cX_t)_{t\geq 0}, \mathbb{Q}_x)\,. \tag{2.1}$$

Lamperti showed that the mapping above is actually one to one. We shall refer to this time change transformation as the Lamperti mapping. We shall use the symbol $\mathbf{E}$ for the expectation operator associated to $\mathbb{Q}$. Finally, let us denote by $T_0^X = \inf\{s \geq 0;\ X_{s-} = 0, X_s = 0\}$ the lifetime of X. Note that $T_0^X = \infty$ or $< \infty$ a.s. according to $E[\zeta_1] \geq 0$ or not. We proceed by stating a general result on a transformation between sspMp.

Proposition 2.1. *Let ζ, with characteristic exponent Υ, be the image via the Lamperti mapping of an α-sspMp X and fix $\beta > 0$. Then, if $\sigma = 0$, the $\frac{\alpha}{\beta}$-sspMp X^β and the α-sspMp $X^{(\beta)}$, obtained as the image via the Lamperti mapping of the Lévy process $\beta\zeta$, have the same image via this mapping. Otherwise, the characteristics exponent of the image via the Lamperti mapping of the $\frac{\alpha}{\beta}$-sspMp X^β is*

$$\Upsilon(\beta\lambda) + \frac{\sigma}{2}\beta(\beta-1)\lambda, \quad \lambda \in \mathbb{R}.$$

The proof is split into the following two lemmas.

Lemma 2.2. *For any $\beta > 0$, we have the following relationship between sspMp,*

$$X_t^{(\beta)} = X^\beta_{\int_0^t X_s^{(\beta)\alpha(\frac{1}{\beta}-1)}ds},\ t < T_0^X, \tag{2.2}$$

where $(X^{(\beta)}, \mathbb{Q})$ is the sspMp associated to the Lévy process $\beta\zeta$.

Proof. Since $\beta > 0$, note that the lifetime of both processes are either finite a.s or infinite a.s. Then, set $A_t = \int_0^t X_s^{-\alpha} ds$ and observe that the Lamperti mapping yields, for any $t < T_0^X$,

$$\begin{aligned}
\log\left(X_t^\beta\right) &= \beta\zeta_{A_t} \\
&= \log\left(X^{(\beta)}_{\int_0^{A_t} e^{\alpha\beta\xi_s} ds}\right) \\
&= \log\left(X^{(\beta)}_{\int_0^t X_s^{\alpha(\beta-1)} ds}\right)
\end{aligned}$$

which completes the proof. □

Next, we recall that the infinitesimal generator, $\mathbf{Q}$, of the α-sspMp X is given for any $\alpha > 0$ and f a smooth function on $\mathbb{R}^+$, by

$$\begin{aligned}
\mathbf{Q}f(x) &= x^{-\alpha}\left(\frac{\sigma}{2}x^2 f''(x) + bxf'(x)\right. \\
&+ \left.\int_{-\infty}^{+\infty}\left((f(e^r x) - f(x)) - xf'(x)r\mathbb{I}_{\{r\leq 1\}}\right)\nu(dr)\right)
\end{aligned}$$

where the three parameters $b \in \mathbb{R}, \sigma \geq 0$ and the measure ν which satisfies the integrability condition $\int_{-\infty}^{\infty}(1 \wedge |r|^2)\,\nu(dr) < +\infty$ form the characteristic triplet of the Lévy process ζ, the image of X by the Lamperti mapping.

Lemma 2.3. *With the notation as above, for any $\beta > 0$, the laplace exponent of the Lévy process associated to the $\frac{\alpha}{\beta}$-sspMp $\left(X^\beta, \mathbb{Q}\right)$ is given by*

$$\Upsilon(\beta\lambda) + \frac{\sigma}{2}\beta(\beta-1)\lambda, \quad \lambda \in \mathbb{R}.$$

Proof. Let us denote by $\mathbf{Q}^\beta$ the infinitesimal generator of $(X^\beta, \mathbb{Q})$ and set $p_\beta(x) = x^\beta$ for any $\beta > 0$. Since the function $x \mapsto p_\beta(x)$ is a homeomorphism of $\mathbb{R}^+$ into itself, we obtain, after some easy computations,

$$\begin{aligned}
\mathbf{Q}^\beta f(x) &= \mathbf{Q}(f \circ p_\beta)(x^{1/\beta}) \\
&= x^{-\alpha/\beta}\left(\frac{\sigma}{2}\beta^2 x^2 f''(x) + \left(b\beta + \frac{\sigma}{2}\beta(\beta-1)\right)xf'(x)\right. \\
&+ \left.\int_{-\infty}^{+\infty}\left((f(e^{\beta r}x) - f(x)) - \beta xf'(x)r\mathbb{I}_{\{r\leq 1\}}\right)\nu(dr)\right).
\end{aligned}$$

The proof is completed by identification. □

The proof of the Proposition follows by putting pieces together.

2.2. **An integral computation.** We proceed by computing an integral, expressed in terms of ratios of gamma functions, which will be very important for the sequel. Indeed, it will be useful for computing the characteristic triplet associated to the families (1.1) and (1.2), which will be provided in Section 3. We fix the following constants

$$\begin{aligned} c &= -\frac{1}{\alpha \cos\left(\frac{\alpha\pi}{2}\right)} > 0 \\ c_\alpha &= \frac{c}{\Gamma(-\alpha)} > 0. \end{aligned}$$

The motivation for the choice of theses constants is given in Remark 3.3 below.

Theorem 2.4. *For any $\alpha, \lambda, \gamma \in \mathbb{C}$, with $1 < \Re(\alpha) < 2$, $\Re(\lambda) > 0$ and $\Re(\alpha + \gamma) > 0$, we have*

$$c_\alpha \int_0^1 \frac{(u^\lambda - 1)u^{\alpha+\gamma-1} - \lambda(u-1)}{(1-u)^{\alpha+1}} du = c\left((\lambda+\gamma)_\alpha - (\gamma)_\alpha\right) - \frac{c_\alpha \lambda}{\alpha - 1}, \tag{2.3}$$

where we recall that $(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)}$ is the Pochhammer symbol and $c_\alpha = \frac{c}{\Gamma(-\alpha)} > 0$.

Proof. In the sequel, we denote by $\mathcal{F}_{\lambda,\alpha,\gamma}$ the integral of the left-hand side on (2.3) divided by c_α. Before starting the proof, we recall the following integral representation of the Beta function, see e.g. Lebedev [21],

$$\begin{aligned} \mathcal{B}(\gamma, \alpha) &= \frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} \\ &= \int_0^1 (1-v)^{\gamma-1} v^{\alpha-1} dv, \quad \Re(\gamma) > 0, \, \Re(\alpha) > 0, \end{aligned}$$

and the recurrence relation for the gamma function $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$. Then, reiteration of integrations by parts yield

$$\mathcal{F}_{\lambda,\alpha,\gamma} = -\frac{\lambda(\lambda+\gamma)_\alpha \Gamma(1-\alpha)}{\alpha(\lambda+\gamma+\alpha-1)} + \frac{\lambda}{\alpha-1} - \frac{\alpha+\gamma-1}{\alpha}\mathcal{F}$$

where we have set

$$\mathcal{F} := \int_0^1 (u^\lambda - 1)u^{\alpha+\gamma-2}(1-u)^{-\alpha} du$$

and we have used the condition $\Re(\alpha+\gamma) > 0$. Next, according to the binomial expansion, we have

$$\begin{aligned}
\mathcal{F} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \int_0^1 (u^\lambda - 1) u^{n+\alpha+\gamma-2} du \\
&= -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \int_0^1 (1-v) v^{\frac{n+\alpha+\gamma-1}{\lambda}-1} dv \\
&= -\lambda \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(n+\alpha+\gamma-1+\lambda)(n+\alpha+\gamma-1)n!} \\
&= -\frac{\lambda}{\alpha+\gamma-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha+\gamma-1)_n}{(\alpha+\gamma)_n n!(n+\alpha+\gamma-1+\lambda)}.
\end{aligned}$$

Using the power series of the ${}_2\mathcal{F}_1$ hypergeometric functions, see e.g. [21, 9.1],

$$ {}_2\mathcal{F}_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n, \quad |\, z \,| < 1, $$

we observe that

$$\begin{aligned}
\mathcal{F} &= -\frac{\lambda}{\alpha+\gamma-1} \lim_{z\to 1^-} \int_0^z u^{\lambda+\alpha+\gamma-2} {}_2\mathcal{F}_1(\alpha, \alpha+\gamma-1; \alpha+\gamma; u) du \\
&= -\frac{\lambda}{\alpha+\gamma-1} \frac{\Gamma(\lambda+\alpha+\gamma-1)}{\Gamma(\lambda+\alpha+\gamma)} \lim_{z\to 1^-} {}_3\mathcal{F}_2\left({}^{\alpha,\alpha+\gamma-1,\lambda+\alpha+\gamma-1}_{\alpha+\gamma,\lambda+\alpha+\gamma}; z\right)
\end{aligned}$$

where the last line follows from [15, 7.512(5)] and ${}_3\mathcal{F}_2$ is the hypergeometric function of degree $(3, 2)$. Note that this later representation holds for $\Re(\alpha) < 2$. We proceed by using a result of Milgram [25, (11)] regarding the limit of the ${}_3\mathcal{F}_2$ function which is

$$\begin{aligned}
\lim_{z\to 1^-} {}_3\mathcal{F}_2\left({}^{\alpha,\alpha+\gamma-1,\lambda+\alpha+\gamma-1}_{\alpha+\gamma,\lambda+\alpha+\gamma}; z\right) &= \frac{-(\alpha+\gamma-1)\Gamma(\alpha+\gamma+\lambda)\Gamma(1-\alpha)}{\lambda\Gamma(\gamma+\lambda)} \\
&+ \frac{(\alpha+\gamma+\lambda-1)\Gamma(\alpha+\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma)\lambda}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{F} &= \frac{\lambda\Gamma(1-\alpha)}{(\gamma+\alpha-1)(\lambda+\gamma+\alpha-1)} \left(\frac{(\gamma+\alpha-1)(\lambda+\gamma)_\alpha}{\lambda} - \frac{(\lambda+\gamma+\alpha-1)(\gamma)_\alpha}{\lambda} \right) \\
&= \Gamma(1-\alpha) \left(\frac{(\lambda+\gamma)_\alpha}{\lambda+\gamma+\alpha-1} - \frac{(\gamma)_\alpha}{(\gamma+\alpha-1)} \right). \qquad (2.4)
\end{aligned}$$

Finally, we obtain

$$ \mathcal{F}_{\lambda,\alpha,\gamma} = \frac{\Gamma(1-\alpha)}{\alpha} \left(-\frac{(\lambda+\gamma)_\alpha(\lambda+\gamma+\alpha-1)}{\lambda+\gamma+\alpha-1} + (\gamma)_\alpha \right) + \frac{\lambda}{\alpha-1} $$

which completes the proof by recalling that $c_\alpha = -c\frac{\Gamma(1-\alpha)}{\alpha}$. □

3. Basic properties of the family $\left(\xi, \mathbb{P}^{(\gamma)}\right)$

Let us denote by $\mathbb{P}^{(\gamma)} = \left(\mathbb{P}_x^{(\gamma)}\right)_{x\in\mathbb{R}}$ the family of probability measures of the process ξ such that $\mathbb{P}_x^{(\gamma)}(\xi_0 = x) = 1$ and recall that the Laplace exponent, denoted by $\psi^{(\gamma)}$, of the process $(\xi, \mathbb{P}^{(\gamma)})$ has the following form

$$\psi^{(\gamma)}(\lambda) \quad = \quad c\left((\lambda+\gamma)_\alpha - (\gamma)_\alpha\right), \quad \lambda \geq 0,$$

where $(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)}$ stands for the Pochhammer symbol and $c = -\frac{1}{\alpha\cos\left(\frac{\alpha\pi}{2}\right)}$, and the parameters α, γ belong to the set

$$\mathcal{K}_{\alpha,\gamma} = \{1 < \alpha < 2, \gamma > -\alpha\}.$$

We denote by $\mathbb{E}_x^{(\gamma)}$ the expectation operator associated to $\mathbb{P}_x^{(\gamma)}$ and write simply $\mathbb{E}^{(\gamma)}$ for $\mathbb{P}_0^{(\gamma)}$. In what follow, we show that it is the Laplace exponent of a spectrally negative Lévy process, we also provide its characteristic triplet and derive some basic properties.

Proposition 3.1. *(1) For $\alpha, \gamma \in \mathcal{K}_{\alpha,\gamma}$, the process $\left(\xi, \mathbb{P}^{(\gamma)}\right)$ is a spectrally negative Lévy process with finite quadratic variation. More specifically, we have*

$$\psi^{(\gamma)}(\lambda) \quad = \quad \tilde{c}_\alpha\lambda + \int_{-\infty}^{0}\left(e^{\lambda y} - 1 - \lambda y\mathbb{I}_{\{|y|<1\}}\right)\nu(dy)$$

where

$$\nu(dy) = c_\alpha\frac{e^{(\alpha+\gamma)y}}{(1-e^y)^{\alpha+1}}dy, \quad y < 0,$$

and

$$\tilde{c}_\alpha \quad = \quad c_\alpha\sum_{k=1}^{\infty}\frac{1}{k(k-\alpha)}\left(\mathcal{B}\left(\frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1\right) - \left(\frac{e-1}{e}\right)^{k-\alpha}\left((\alpha+\gamma-1)e^{1-\alpha-\gamma} - 1\right)\right)$$

where $\mathcal{B}(.;.,.)$ stands for the incomplete Beta function.

(2) The random variable $\left(\xi_1, \mathbb{P}^{(\gamma)}\right)$ admits negative exponential moments of order lower than $\gamma+\alpha$, i.e. for any $\lambda < \gamma+\alpha$, we have

$$\mathbb{E}^{(\gamma)}\left[e^{-\lambda\xi_1}\right] < +\infty.$$

(3) We have the following invariance property by Girsanov transform, i.e. for any $\gamma \in \mathcal{K}_{\alpha,\gamma}$,

$$d\mathbb{P}_0^{(\gamma)}{}_{|F_t} = e^{\gamma\xi_t - \psi^{(0)}(\gamma)t}d\mathbb{P}_0^{(0)}{}_{|F_t}, \quad t > 0. \tag{3.1}$$

(4) *The first moments have the following expressions*

$$\mathbb{E}^{(\gamma)}[\xi_1] = c(\gamma)_\alpha(\Psi(\gamma+\alpha)-\Psi(\gamma)), \gamma > -\alpha,$$

$$\mathbb{E}^{(0)}[\xi_1] = c\Gamma(\alpha) \tag{3.2}$$

$$\mathbb{E}^{(1-\alpha)}[\xi_1] = c\frac{1}{\Gamma(1-\alpha)}(-E_\gamma - \Psi(1-\alpha)) \tag{3.3}$$

$$\mathbb{E}^{(-1)}[\xi_1] = -c\Gamma(\alpha-1) \tag{3.4}$$

where $\Psi(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$ *is the digamma function and* E_γ *stands for Euler-Mascheroni constant.*

(5) *For any* $1 < \alpha < 2$*, there exits* γ_α*, with* $-\alpha < \gamma_\alpha < 0$*, such that* $\mathbb{E}^{(\gamma_\alpha)}[\xi_1] = 0$*. Moreover, for any* $\gamma < \gamma_\alpha$ *the Cramér condition holds, i.e. for any* $\gamma < \gamma_\alpha$*, there exists* $\lambda_\alpha > 0$ *such that*

$$\mathbb{E}^{(\gamma)}\left[e^{\lambda_\alpha \xi_1}\right] = 1. \tag{3.5}$$

(6) *Consequently, for any fixed* $1 < \alpha < 2$*, the process* $\left(\xi_t, \mathbb{P}^{(\gamma_\alpha)}\right)$ *oscillates and otherwise* $\lim_{t\to\infty}\left(\xi_t, \mathbb{P}^{(\gamma)}\right) = sgn(\gamma-\gamma_\alpha)\infty$ *a.s.*

(7) *For* $\gamma = 0$ *or* -1*, the scale function of* $\left(\xi, \mathbb{P}^{(\gamma)}\right)$ *is given by*

$$\mathcal{W}^{(\gamma)}(x) = \frac{1}{\Gamma(\alpha)}e^{-\gamma x}(1-e^{-x})^{\alpha-1}, \quad x > 0.$$

(8) *Finally, we have the following two limits results:*

(a) *For any* $\gamma \in \mathcal{K}_{\alpha,\gamma}$*, the process* $\left(\xi, \mathbb{P}^{(\gamma)}\right)$ *converges in distribution as* $\alpha \to 2$ *to a Brownian motion with drift* $\gamma + \frac{1}{2}$*.*

(b) *For any fixed* $1 < \alpha < 2$*, the process* $\left(\lambda^{\frac{1}{\alpha}}\xi_{\lambda t}, \mathbb{P}^{(0)}\right)$ *converges in distribution as* $\lambda \to \infty$ *to a spectrally negative* α*-stable process.*

Proof. (1) First, from (2.3), we deduce that

$$\begin{aligned}
\psi^{(\gamma)}(\lambda) &= c\left((\lambda+\gamma)_\alpha - (\gamma)_\alpha\right) \\
&= \frac{c_\alpha\lambda}{1-\alpha} + \int_0^1 \left((u^\lambda - 1)u^{\alpha+\gamma-1} - \lambda(u-1)\right)\frac{c_\alpha du}{(1-u)^{\alpha+1}} \\
&= \left(c_\alpha\left(\int_0^1 \frac{(\log(u)-(u-1))\mathbb{I}_{\{|\log(u)|<1\}}}{(1-u)^{\alpha+1}}du + \int_{-\infty}^{\frac{1}{e}}(1-u)^{-\alpha}du\right)\right. \\
&+ \left. c_\alpha\int_0^1 \frac{\left(u^{\alpha+\gamma-1}-1\right)\log(u)\mathbb{I}_{\{|\log(u)|<1\}}}{(1-u)^{\alpha+1}}du\right)\lambda \\
&+ \int_0^1 \left(u^\lambda - 1 - \lambda\log(u)\mathbb{I}_{\{|\log(u)|<1\}}\right)\frac{c_\alpha u^{\alpha+\gamma-1}du}{(1-u)^{\alpha+1}} \\
&= \tilde{c}_\alpha\lambda + \int_0^\infty \left(e^{\lambda y} - 1 - \lambda y\mathbb{I}_{\{|y|<1\}}\right)\frac{c_\alpha e^{(\alpha+\gamma)y}du}{(1-e^y)^{\alpha+1}}.
\end{aligned}$$

where we have set

$$\tilde{c}_\alpha = c_\alpha \left(\int_0^1 \frac{(\log(u) - (u-1))\mathbb{I}_{\{|\log(u)|<1\}}}{(1-u)^{\alpha+1}} du + \int_{-\infty}^{\frac{1}{e}} (1-u)^{-\alpha} du \right.$$
$$\left. + \int_0^1 \frac{\left(u^{\alpha+\gamma-1} - 1\right) \log(u)\mathbb{I}_{\{|\log(u)|<1\}}}{(1-u)^{\alpha+1}} du \right)$$

Hence, we recognize the Lévy-Khintchine representation of a one-sided Lévy process. Moreover, the quadratic finite variation property follows from the following asymptotic behavior of the Pochhammer symbol, see e.g. [21],

$$(3.6)\ (z+\gamma)_\alpha^{-1} \quad = \quad z^{-\alpha}\left[1 + \frac{(-\alpha)(2\gamma - \alpha - 1)}{2z} + O(z^{-2})\right],\ |\arg z| < \pi - \delta,\ \delta > 0,$$

and the condition $1 < \alpha < 2$. It remains to compute the constant $\tilde{c}_\alpha$. Performing the change of variable $v = 1 - u$, we get, for the first term on the left-hand side of the previous identity,

$$\begin{aligned} \int_0^{\frac{e-1}{e}} (\log(1-v) + v)\frac{c_\alpha}{v^{\alpha+1}} dv &= -c_\alpha \sum_{k=2}^\infty \frac{1}{k} \int_0^{\frac{e-1}{e}} v^{k-\alpha-1} dv \\ &= -c_\alpha \sum_{k=2}^\infty \frac{1}{k(k-\alpha)} \left(\frac{e-1}{e}\right)^{k-\alpha}. \end{aligned}$$

Moreover, proceeding as above, we have

$$\begin{aligned} &\int_0^1 \frac{\left(u^{\alpha+\gamma-1} - 1\right) \log(u)\mathbb{I}_{\{|\log(u)|<1\}}}{(1-u)^{\alpha+1}} du \\ = \ & -c_\alpha \sum_{k=1}^\infty \frac{1}{k} \int_0^{\frac{e-1}{e}} v^{k-\alpha-1}((1-v)^{\alpha+\gamma-1} - 1) dv \\ = \ & c_\alpha \sum_{k=1}^\infty \frac{\alpha+\gamma-1}{k(k-\alpha)} \left(\int_0^{\frac{e-1}{e}} v^{k-\alpha}(1-v)^{\alpha+\gamma-2} dv \right. \\ & \left. - \left(\frac{e-1}{e}\right)^{k-\alpha} e^{1-\alpha-\gamma} \right) \\ = \ & c_\alpha \sum_{k=1}^\infty \frac{\alpha+\gamma-1}{k(k-\alpha)} \left(\mathcal{B}\left(\frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1\right) - \right. \\ & \left. \left(\frac{e-1}{e}\right)^{k-\alpha} e^{1-\alpha-\gamma} \right). \end{aligned}$$

Putting pieces together, one gets

$$\begin{aligned}\tilde{c}_\alpha &= c_\alpha \sum_{k=1}^{\infty} \frac{1}{k(k-\alpha)} \left(\mathcal{B}\left(\frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1\right) - \right. \\ &\left. \left(\frac{e-1}{e}\right)^{k-\alpha} \left((\alpha+\gamma-1)e^{1-\alpha-\gamma} - 1\right) \right)\end{aligned}$$

which completes the description of the characteristic triplet of $\left(\xi, \mathbb{P}^{(\gamma)}\right)$.

(2) This item follows from the fact that the mapping $\lambda \to \psi^{(\gamma)}(\lambda)$ is well defined on $(-\gamma+\alpha, \infty)$.

(3) It is simply the Esscher transform.

(4) The expressions for the first moment of $\left(\xi_1, \mathbb{P}^{(\gamma)}\right)$ is obtained from the formula

$$\frac{\partial}{\partial \lambda}(\lambda+\gamma)_\alpha = (\lambda+\gamma)_\alpha(\Psi(\lambda+\gamma+\alpha) - \Psi(\lambda+\gamma)). \tag{3.7}$$

Moreover, for $\gamma = 0$ and $\gamma = -1$ we use the recurrence formulae for the digamma function, $\Psi(u+1) = \frac{1}{u} + \Psi(u)$, see [21, Formula 1.3.3] and for the Gamma function.

(5) Since, for any $\gamma \in \mathcal{K}_{\alpha,\gamma}$, the mapping $\lambda \mapsto \psi^{(\gamma)}(\lambda)$ is convex and continuously differentiable on $(\alpha-\gamma, \infty)$, its derivative is continuous and increasing. Hence, the mapping $\gamma \mapsto \psi^{(\gamma)}(0^+)$ is continuous and increasing on $\mathcal{K}_{\alpha,\gamma}$. Moreover, noting that $\mathbb{E}^{(-1)}[\xi_1] < 0 < \mathbb{E}^{(0)}[\xi_1]$ for any $1 < \alpha < 2$, we deduce that for each $1 < \alpha < 2$, there exists an unique $-1 < \gamma_\alpha < 0$ such that $\mathbb{E}^{(\gamma_\alpha)}[\xi_1] = 0$. Since for any $\gamma < \gamma_\alpha$, $\mathbb{E}^{(\gamma)}[\xi_1]$ is negative and $\lambda \mapsto \psi^{(\gamma)}(\lambda)$ is convex with $\lim_{\lambda\to\infty} \psi^{(\gamma)}(\lambda) = \infty$, we deduce that there exists $\lambda_\alpha > 0$ such that $\psi^{(\gamma)}(\lambda_\alpha) = 0$.

(6) The long time behavior of $\left(\xi, \mathbb{P}^{(\gamma)}\right)$ follows from the previous item and the strong law of large numbers.

(7) The expression of the scale function is derived from the following identity, see [15, 3.312,1.], for $\Re(\alpha), \Re(\lambda+\gamma) > 0$,

$$\int_0^\infty e^{-(\lambda+\gamma)x}(1-e^{-x})^{\alpha-1} dx = \frac{\Gamma(\lambda+\gamma)\Gamma(\alpha)}{\Gamma(\lambda+\gamma+\alpha)}.$$

(8) It is enough to show that the random variable $\left(\xi_1, \mathbb{P}^{(\gamma)}\right)$ converges in law to a normal distribution. By continuity of the function $\psi^{(\gamma)}(\lambda)$ in α, we get the result after easy manipulation of the Gamma function. Similarly, for the second limit, we simply need to show that the random variable $\left(\xi_1, \mathbb{P}^{(0)}\right)$ belong to the domain of attraction of a stable distribution, i.e. $\lim_{\eta\to\infty} \eta\psi^{(0)}(\eta^{-\frac{1}{\alpha}}\lambda) = c\lambda^\alpha$. The claim follows by means of the asymptotic of the ratio of Gamma functions, see (3.6).

□

Remark 3.2. The choice of the constant c_α is motivated by the item (8b). Actually, the coefficients of the Laplace exponent of a completely asymmetric stable random variable is given for any $\iota > 0$ by $c(\iota) = -\frac{\iota^\alpha}{\alpha\cos\left(\frac{\alpha\pi}{2}\right)} > 0$, see e.g. [34, Proposition 1.2.12]. Thus, we have made the choice $\iota^\alpha = 1$.

Remark 3.3. Note that Caballero and Chaumont [8] show, by characterizing its characteristic triplet, that the process $(\xi, \mathbb{P}^{(0)})$ is the image via the Lamperti mapping of a spectrally negative regular α-stable process conditioned to stay positive. For any $\gamma \in \mathcal{K}_{\alpha,\gamma}$, the expression of the Laplace exponent appears as an example but without proof, in [29].

4. **Law of some exponential functionals via**...

In this section, we derive the explicit law of the exponential functionals associated to some elements of the family of Lévy processes introduced in this paper and to some transforms of these elements.

We proceed by defining a family of functions which will appear several times in the remaining part of the paper. The Wright hypergeometric function is defined as, see [12, 1,Section 4.1],

$$ {}_p\Psi_q\left(\begin{array}{c} (A_1,a_1),\dots(A_p,a_p) \\ (B_1,b_1)\dots(B_q,b_q) \end{array} \middle| \, z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(A_i n + a_i)}{\prod_{i=1}^{q}\Gamma(B_i n + b_i)} \frac{z^n}{n!} $$

where p, q are nonnegative integers, $a_i \in \mathbb{C}$ $(i = 1\dots p), b_j \in \mathbb{C}$ $(j = 1\dots q)$ and the coefficients $A_i \in \mathbb{R}^+$ $(i = 1\dots p)$ and $B_j \in \mathbb{R}^+$ $(j = 1\dots q)$ are such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$. Under such conditions, it follows from the following asymptotic formula of ratio of gamma functions, see (3.6),

$$ \frac{\Gamma(z+\gamma)}{\Gamma(z+\gamma+\alpha)} = z^{-\alpha}\left[1 + \frac{(-\alpha)(2\gamma-\alpha-1)}{2z} + O(z^{-2})\right], \quad |\arg z| < \pi - \delta, $$

that ${}_p\Psi_q(z)$ is an entire function with respect to z. We postpone to Section 6 for a more detailed description of this class of function.

4.1. ... **an important continuous state branching process.**

In this part, we establish a connection between the dual of an element of the family of Lévy processes introduced in this paper and a self-similar continuous state branching process. As a byproduct, we derive the explicit law of the exponential functional associated to the dual of this element.

To this end, we recall that continuous state branching processes form a class of non negative valued Markov processes which appear as limit of integer valued branching processes. Lamperti [19] showed that when the units and the initial state size is allowed to tend to infinity, the limiting process is necessarily a self-similar continuous state branching process with index lower than 1. We denote this process by $(Y, \mathbb{Q}^{(0)})$, i.e. $\mathbb{Q}^{(0)} = (\mathbb{Q}^{(0)})_{x>0}$ is a family of probability measures such that $\mathbb{Q}^{(0)}_x(Y_0 = x) = 1$. The associated expectation operator is $\mathbf{E}^{(0)}$. Moreover, Lamperti showed that the semi-group of $(Y, \mathbb{Q}^{(0)})$ is characterized by its spatial Laplace transform as follows, for any $\lambda, x \geq 0$,

$$ \mathbf{E}^{(0)}_x\left[e^{-\lambda Y_t}\right] = e^{-xd\lambda(1+ct\lambda^{\kappa})^{-1/\kappa}}, \tag{4.1} $$

for some $0 < \kappa \leq 1$ and some positive constants d, c. On the other hand, he also observed in [18] that a continuous state branching process can be obtained from a *spectrally positive* Lévy process, ζ, by a time change. More precisely, consider ζ started at $x > 0$ and write

$$ T_0^{\zeta} = \inf\{s \geq 0;\, \zeta_s = 0\}. $$

Next, let

$$\Lambda_t = \int_0^t \zeta_s^{-1} ds, \quad t < T_0^\zeta$$

and

$$V_t = \inf\{s \geq 0;\ \Lambda_s > t\} \wedge T_0^\zeta.$$

Then, the time change process $\zeta \circ V$ is a continuous state branching process starting at x and the Laplace exponent φ of ζ is called the branching mechanism. Finally, we recall that the law of the absorption time

$$i = \inf\{s \geq 0; \zeta \circ V_s = 0\}$$

has been computed explicitly by Grey [16]. More specifically, put $\phi(0) = \inf\{s \geq 0; \varphi(s) > 0\}$ (with the usual convention that $\inf \emptyset = \infty$) and assume that $\int^\infty \varphi^{-1}(s)ds < \infty$ and $\phi(0) < \infty$, then

(4.2) the law of $g(i)$ for $\zeta \circ V$ starting at $x > 0$ is exponential with parameter x,

where $g : (0, \infty) \to (\phi(0), \infty)$ is the inverse mapping of $\int_t^\infty \varphi^{-1}(s)ds$. We are now ready to state and proof the main result of this part.

Theorem 4.1. *For any $0 < \kappa \leq 1$, we have*

$$\left(\int_0^\infty e^{-\kappa \xi_s} ds, \mathbb{P}_0^{(0)} \right) \sim cG(1)^{-\kappa}$$

where $G(1)$ is a Gamma random variable of parameter 1.

In the case $\kappa = 1$, $\left(\xi, \mathbb{P}^{(0)}\right)$ is a Brownian motion with positive drift $\frac{1}{2}$, see item 8a in Proposition 3.1, and the result above corresponds to Dufresne's result [11]. We split the proof in several lemmas. First, let us denote by X a *spectrally positive* α-stable Lévy process killed when it hits zero. It is plain that it is also a α-sspMp. We have the following.

Lemma 4.2. *The Lévy process associated, via the Lamperti mapping, to X is the dual, with respect to the Lebesgue measure, of $(\xi, \mathbb{P}^{(0)})$, i.e. $(-\xi, \mathbb{P}^{(0)})$.*

Proof. First, by Hunt switching identity, see e.g. Getoor and Sharpe [14], we have that X is in duality, with respect to the Lebesgue measure, with the spectrally negative α-stable Lévy process killed when entering the negative real line. Moreover, it is well known, see e.g. Bertoin [2] that the spectrally negative α-stable Lévy process conditioned to stay positive, denoted by $\widehat{X}^\uparrow$, is obtained as h-transform, in the Doob sense, of the killed process with $h(x) = x^{\alpha-1}, x > 0$. Thus, it is plain that $\widehat{X}^\uparrow$ is the dual, with respect to the reference measure $y^{\alpha-1}dy$, of X. Moreover, the Lévy process associated via the Lamperti mapping to $\widehat{X}^\uparrow$ is $(\xi, \mathbb{P}^{(0)})$, see Remark 3.3. Since X and $\widehat{X}^\uparrow$ are sspMp, the conclusion follows from Bertoin and Yor [4]. $\square$

Lemma 4.3. *The spectrally positive Lévy process associated, via the Lamperti mapping, to the κ-ssMp $(Y, \mathbb{Q}^{(0)})$ is $(-\xi, \mathbb{P}^{(0)})$.*

Remark 4.4. As mentioned above, in the case $\kappa = 1$, $(-\xi, \mathbb{P}^{(0)})$ corresponds to a Brownian motion with drift $-\frac{1}{2}$, then $(2Y, \mathbb{Q}^{(0)})$ is a squared Bessel process of dimension 0 and we recover the well-known formula, see e.g. Revuz and Yor [31],

$$\mathbf{E}_x^{(0)}\left[e^{-2\lambda Y_t}\right] = e^{-x\lambda(1+t2\lambda)^{-1}}.$$

Proof. First, we show that the branching mechanism associated to $(Y, \mathbb{Q}^{(0)})$ is the Laplace exponent of a spectrally positive α-stable process. It is likely that this result is known. Since we did not find any reference and for sake of completeness we provide an easy proof. It is well-know, see e.g. Zeng-Hu [38], that the semi-group of a continuous state branching process with branching mechanism φ admits as spatial Laplace transform the following expression, for any $\lambda \geq 0$,

$$e^{-x\vartheta_\lambda(t)}$$

where $\vartheta : [0,\infty) \to [0,\infty)$ solves the following boundary valued differential equation

$$\vartheta'_\lambda(t) = -\varphi(\vartheta_\lambda(t)), \quad \vartheta_\lambda(0) = \lambda.$$

It is then not difficult to check, in the case $\varphi(\lambda) = \frac{c}{\kappa}\lambda^\alpha$, that

$$\vartheta_\lambda(t) = \lambda\,(1 + ct\lambda^\kappa)^{-1/\kappa}$$

which is the Laplace exponent of $(Y, \mathbb{Q}^{(0)})$ given in (4.1) with $d = 1$ and $0 < \kappa = \alpha - 1 \leq 1$. Then, we deduce that $(Y, \mathbb{Q}^{(0)})$ is obtained from X by the random time change described above. Finally, from Lemma 4.2 and Lemma 2.2, by choosing β_α such that $\beta_\alpha = -\frac{\beta_\alpha}{\alpha(\beta_\alpha - 1)}$, i.e. $\beta_\alpha = \frac{\kappa}{\alpha}$, we get that the image via the Lamperti mapping of $(Y^{\beta_\alpha}, \mathbb{Q}^{(0)})$ is $(\beta_\alpha\xi,\ \mathbb{P}^{(0)})$. By using Lemma 2.3 and observing that $(Y, \mathbb{Q}^{(0)})$ is κ-self-similar we complete the proof. □

The proof of the Theorem 4.1 follows readily by observing that for $\varphi(\lambda) = \frac{c}{\kappa}\lambda^\alpha$, $g(t) = (ct)^{\frac{1}{1-\alpha}}$ and from the identity $(i, \mathbb{Q}_x^{(0)}) \overset{(d)}{=} (x^\kappa \int_0^\infty e^{-\kappa\xi_s}ds, \mathbb{P}_0^{(0)})$. In the following sub-Section, we shall provide the expression of the semi-group of $(Y, \mathbb{Q}^{(0)})$ in terms of a power series. We conclude this part by providing the Laplace transform of the first passage time below for the continuous state branching process $(Y, \mathbb{Q}^{(0)})$. That is for the stopping time

$$T_a^Y \quad = \quad \inf\{s \geq 0;\ Y_s = a\}.$$

Proposition 4.5. *For any $x > a > 0$, we have*

$$\mathbf{E}_x^{(0)}\left[e^{-qT_a^Y}\right] = \frac{\widehat{\mathcal{I}}(q^{\frac{1}{\kappa}}c\kappa x)}{\widehat{\mathcal{I}}(q^{\frac{1}{\kappa}}c\kappa a)}, \quad q \geq 0.$$

where

$$\begin{aligned}
\widehat{\mathcal{I}}(x) &= x\int_0^\infty e^{-t-xt^{-\frac{1}{\kappa}}}t^{-\frac{1}{\kappa}-1}dt \\
&= \sum_{n=0}^\infty (-1)^n \frac{\Gamma(1-\frac{n}{\kappa})}{n!}x^n + \kappa\sum_{n=0}^\infty (-1)^n \frac{\Gamma(\kappa(n+1))}{n!}x^{\kappa(n+1)} \\
&= {}_1\Psi_0\left(\ (-\tfrac{1}{\kappa}, 1)\ \middle|\ -x\right) + \kappa\,{}_1\Psi_0\left(\ (\kappa,\kappa)\ \middle|\ -x\right)
\end{aligned}$$

Proof. First, from (4.2), we deduce readily the identity

$$\mathbf{E}_x^{(0)}\left[e^{-qT_0^Y}\right] = \widehat{\mathcal{I}}(q^{\frac{1}{\kappa}}c\kappa x), \quad q \geq 0.$$

The first part of the claims is completed by an application of the strong Markov property and using the fact that $(Y, \mathbb{Q}^{(0)})$ has no negative jumps. To get the expression of the integral as a power series we follow a line of reasoning similar to Neretin [27]. First, consider the space $L^2(\mathbb{R}^+, \frac{dx}{x})$ and denote by $I_{\varrho,h}(x,v)$ for $\varrho < 0$ and $\Re(x), \Re(v), \Re(h) > 0$ the inner product of the functions $e_x^{\varrho}(t) = e^{-xt^{\varrho}}$ and $e_{v,h}(t) = t^h e^{-vt^{\varrho}}$, i.e.

$$I_{\varrho,h}(x,v) \quad = \quad \int_0^\infty e^{-vt-xt^{\varrho}} t^{h-1} dt.$$

Then, the Mellin transform of e_x^{ϱ} is

$$\begin{aligned}
\tilde{e}_x^{\varrho}(\lambda) &= \int_0^\infty e^{-xt^{\varrho}} t^{\lambda-1} dt \\
&= \frac{\operatorname{sgn}(\varrho)}{\varrho} \int_0^\infty e^{-xu} u^{\lambda/\varrho-1} du \\
&= \frac{\operatorname{sgn}(\varrho)\Gamma(\lambda/\varrho)}{\varrho x^{\lambda/\varrho}}.
\end{aligned}$$

While the Mellin transform of $e_x^{v,h}$ is

$$\tilde{e}_x^{v,h}(\lambda) \quad = \quad v^{-h+\lambda}\Gamma(h).$$

By the Plancherel formula for the Mellin transform, we have

$$\int_0^\infty e_x^{\varrho}\overline{e_x^{v,h}}\frac{dx}{x} \quad = \quad \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{e}_x^{v,h}(i\lambda)\overline{\tilde{e}_x^{v,h}(i\lambda)} d\lambda$$

that is

$$I_{\varrho,h}(x,v) \quad = \quad \frac{\operatorname{sgn}(\varrho)}{2\pi\varrho v^h}\int_{-\infty}^\infty \Gamma(\lambda/\varrho)\Gamma(h+i\lambda)u^{-i\lambda/\varrho}v^{i\lambda} d\lambda.$$

Then we perform the change of variable $z = i\lambda/\varrho$ and consider an arbitrary contour $\mathfrak{C}$ coinciding with the imaginary axis near $\pm\infty$ and leaving all the poles of the integrand of the left side. We get the series expansions by summing the residues and by choosing $h = v = 1$. $\square$

4.2. ... **a family of continuous state branching processes with immigration.** We recall that $\kappa = \alpha - 1$, and for any $\delta \in \mathbb{R}^+$ we write $\mathbb{P}^{(0,\delta)} = \left(\mathbb{P}_x^{(0,\delta)}\right)_{x\in\mathbb{R}}$ for the family of probability measures of the Lévy process ξ, which admits the following Laplace exponent

$$\begin{aligned}
\psi^{(0,\delta)}(\lambda) &= \psi^{(0)}(\lambda) - \frac{\alpha\delta}{\lambda+\kappa}\psi^{(0)}(\lambda), \quad \lambda \geq 0 \\
&= c\,(\lambda+\kappa-\alpha\delta)\,\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}.
\end{aligned}$$

We start by computing the characteristic triplet of $(\xi, \mathbb{P}^{(0,\delta)})$.

Proposition 4.6. *The characteristic triplet of* $(\xi, \mathbb{P}^{(0,\delta)})$ *is given by* $\sigma^{(\delta)} = 0$,

$$b^{(\delta)} \quad = \quad c\Gamma(\alpha)\left(1 - \frac{\alpha\delta}{\kappa}\right),$$

and

$$\nu^{(\delta)}(dy) \quad = \quad c_\alpha \frac{e^{\alpha y}}{(1-e^y)^{\alpha+1}}\left(1 + \delta(e^{-y} - 1)\right) dy, \quad y < 0.$$

where

$$\log\left(\mathbb{E}^{(0,\delta)}\left[e^{-\lambda \xi_1}\right]\right) = b^{(\delta)}\lambda + \int_{-\infty}^{0}\left(e^{\lambda y} - 1 - \lambda y\right)\nu^{(\delta)}(dy).$$

Proof. First, writing simply here ψ for $\psi^{(0)}$, we have from (2.4) by choosing $\gamma = 0$ that

$$\begin{aligned}
\frac{\alpha\delta}{\lambda+\kappa}\psi(\lambda) &= \frac{c\alpha\delta}{\lambda+\kappa}\frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)} \\
&= \frac{c\alpha\delta}{\Gamma(1-\alpha)}\int_0^1\left(u^\lambda - 1\right)u^{\alpha-2}(1-u)^{-\alpha}du \\
&= -\delta\int_{-\infty}^0\left(e^{\lambda y} - 1\right)\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^\alpha} \\
&= -\delta\left(\int_{-\infty}^0\left(e^{\lambda y} - 1 - \lambda y\right)\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^\alpha} + \lambda\int_{-\infty}^0 y\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^\alpha}\right) \\
&= -\delta\int_{-\infty}^0\left(e^{\lambda y} - 1 - \lambda y\right)\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^\alpha} + \lambda c\Gamma(\alpha)\frac{\alpha\delta}{\kappa}.
\end{aligned}$$

where we have used the identities $c_\alpha = \frac{c}{\Gamma(-\alpha)} > 0$ and

$$\begin{aligned}
\int_{-\infty}^0 y\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^\alpha} &= -\lim_{\lambda\to 1}\frac{\partial}{\partial\lambda}\int_{-\infty}^0\left(e^{\lambda y} - 1 - \lambda y\right)\frac{c_\alpha e^{\kappa y}dy}{(1-e^y)^{\alpha+1}} \\
&= -\lim_{\lambda\to 1}\frac{\partial}{\partial\lambda}\psi^{(-1)}(\lambda) - c\frac{\Gamma(\alpha)}{\kappa} \\
&= -\psi'(0) - c\frac{\Gamma(\alpha)}{\kappa}.
\end{aligned}$$

The proof is completed by putting pieces together. □

Next, set $\delta_\kappa = \frac{\delta}{\kappa}$ and $M_\delta = \mathbb{E}^{(0,\delta)}[\xi_1]$ and note from the previous proposition that

$$M_\delta \quad = \quad c\Gamma(\alpha)\left(1 - \alpha\delta_\kappa\right).$$

In particular, we have $M_\delta < 0$ if $\delta > \frac{\kappa}{\alpha}$. Under such a condition, we write simply

$$\left(\Sigma_\infty, \mathbb{P}_0^{(0,\delta)}\right) = \left(\int_0^\infty e^{\kappa\xi_s}ds, \mathbb{P}_0^{(0,\delta)}\right).$$

We are now ready to state to the main result of this section.

Theorem 4.7. *Let $0 < \kappa < 1$. Then, for any $\delta > \frac{\kappa}{\alpha}$, the law of the positive random variable $(\Sigma_\infty, \mathbb{P}_0^{(0,\delta)})$ is absolutely continuous with an infinitely continuously differentiable density, denoted by $f_\infty^{(\delta)}$, and*

$$\begin{aligned} f_\infty^{(\delta)}(y) &= M_\delta y^{-\alpha\delta} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha\delta_\kappa)}{n!\Gamma(\kappa n+\alpha\delta)} y^{-n} \\ &= M_\delta y^{-\alpha\delta} {}_1\Psi_1 \left(\begin{array}{c} (1,\alpha\delta_\kappa) \\ (\kappa,\alpha\delta) \end{array} \middle| -y^{-1} \right),\ y > 0. \end{aligned}$$

We deduce, from Section 6, the following asymptotic behaviors

$$\begin{aligned} f_\infty^{(\delta)}(y) &\sim M_\delta \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha\delta_\kappa)}{n!\Gamma(-\kappa n)} y^n \quad \text{as } y \to 0. \\ f_\infty^{(\delta)}(y) &\sim M_\delta \frac{\Gamma(\alpha\delta_\kappa)}{\Gamma(\alpha\delta)} y^{-\alpha\delta} \quad \text{as } y \to \infty. \end{aligned}$$

Remark 4.8. (1) For $\delta > \frac{\kappa}{\alpha}$, the law of $\left(\Sigma_\infty, \mathbb{P}^{(0,\delta)}\right)$ is a generalization of the inverse Gamma distribution. Indeed, specifying on $\kappa = 1$, i.e. $\alpha = 2$, $c = \frac{1}{2}$ and $\delta > \frac{1}{2}$, the expression above reduces to

$$f_\infty^{(\delta)}(y) = 2\frac{y^{-2\delta}}{\Gamma(2\delta-1)} e^{-\frac{1}{y}}$$

which corresponds to Dufresne's result [11], i.e.

$$\int_0^\infty e^{B_s - (\delta - \frac{1}{2})s} ds \sim \frac{1}{2G(2\delta-1)}$$

where we recall that $G(\delta)$ is a Gamma random variable with parameter $\delta > 0$.

(2) Note also that, the densities $M_\delta^{-1} f_\infty^{(\delta)}$ converges, as $\delta \to \frac{\kappa}{\alpha}$ to a density probability distribution given by

$$f^{(\frac{\kappa}{\alpha})}(y) = y^{-1} \sum_{n=0}^{\infty} \frac{(-y)^{-n}}{\Gamma(\kappa(n+1))},\ y > 0, \tag{4.3}$$

which is the inverse of a positive Linnik law of parameters (κ, κ), see [22].

The proof of the Theorem above in split in several intermediate results which we find worth being stated. The plan of the proof is as follows. We first characterize, in terms of their spatial Laplace transform the class of continuous state branching processes with immigration (for short *cbip*) which enjoy the scaling property. On the one hand, by inverting this Laplace transform, we provide an expression for the density of the semi-group of this family. In particular, we shall obtain the expression of the density of their entrance laws at 0. On the other hand, we shall characterize the Lévy processes associated to this family of sspMp via the Lamperti mapping. Finally, we shall show how to relate the density of these entrance laws to the density of the law of the exponential functionals under study.

We start with the following easy result which gives a complete characterization, in terms of their Laplace transforms, of self-similar cbip. To this end, we now recall the definition

of a cbip with parameters $[\varphi, \chi]$. It is well known, see e.g. [38], that the semi-group of a cbip with branching mechanism φ and immigration mechanism χ, where χ is the Laplace exponent of a positive infinitely divisible random variable, admits as a spatial Laplace transform the following expression

$$\exp\left(-x\vartheta_\lambda(t) - \int_0^t \chi(\vartheta_\lambda(s))ds\right), \qquad x, t \geq 0. \tag{4.4}$$

Lemma 4.9. *A cbip is self-similar of index κ if and only if $0 < \kappa \leq 1$ and it corresponds to the cbip with parameters $[\varphi, \delta\chi]$ where $\delta > 0$, $\varphi(\lambda) = \frac{c}{\kappa}\lambda^{\kappa+1}$ and $\chi(\lambda) = c(\frac{\kappa+1}{\kappa})\lambda^\kappa$. Its Laplace transform has the following expression, for $\delta, x, \lambda > 0$*

$$\mathbf{E}_x^{(\delta)}\left[e^{-\lambda Y_t}\right] = \Lambda_t^{(\delta)}(\lambda, x)$$

where

$$\Lambda_t^{(\delta)}(\lambda, x) \quad = \quad (1 + ct\lambda^\kappa)^{-\alpha\delta_\kappa}\, e^{-x\lambda(1+ct\lambda^\kappa)^{-1/\kappa}}.$$

In particular its entrance law is characterized by

$$\mathbf{E}_{0^+}^{(\delta)}\left[e^{-\lambda Y_t}\right] = (1 + ct\lambda^\kappa)^{-\alpha\delta_\kappa}. \tag{4.5}$$

We denote this family of processes by $(Y, \mathbb{Q}^{(\delta)})_{\delta>0}$.

Proof. The sufficient part follows readily from the definition of the cbip and by observing that for any $a > 0$, $\Lambda_{a^\kappa t}^{(\delta)}(\lambda, ax) = \Lambda_t^{(\delta)}(a\lambda, x)$. The necessary part follows from the fact that the unique self-similar branching process has its Laplace transform given by (4.1) and thus the immigration has to satisfy the self-similarity property. Since we have for any $a > 0$, $a\vartheta_\lambda(a^\kappa t) = \vartheta_{a\lambda}(t)$, we need that

$$\chi(a\vartheta_{a\lambda}(t)) = a^\kappa \chi(\vartheta_{a\lambda}(\kappa t))$$

which is only possible for $\chi(a) = Ca^\kappa$ for some positive constant C (since χ is the Laplace exponent of a subordinator). The claim follows. $\square$

Remark 4.10.
(1) We mention that $(Y, \mathbb{Q}^{(1)})$ corresponds to the continuous state branching process $(Y, \mathbb{Q}^{(0)})$ conditioned to never extinct in the terminology of Lambert [17]. Indeed, he showed that the latter corresponds to the cbpi with immigration φ'.
(2) The Laplace transform of the entrance law (4.5) appears in a paper of Pakes [28] where he studies scaled mixtures of (symmetric) stable laws. More precisely, denoting by $Y_1^{(\delta)}$ the entrance law at time 1 of $(Y, \mathbb{Q}^{(\delta)})$, we have the following identity in law

$$Y_1^{(\delta)} \stackrel{(d)}{=} G(\delta)^\kappa S_\kappa,$$

where S_κ is a positive stable law of index κ and the two random variables on the right-hand side are considered to be independent.

We proceed by providing the expression of the semi-group of $(Y, \mathbb{Q}^{(\delta)})$.

Proposition 4.11. *For any $\delta > 0$, the semi-group of $(Y, \mathbb{Q}^{(\delta)})$ admits a density, with respect to the Lebesgue measure, denoted by $p_t^{(\delta)}(.,.)$, which is given for any $x, y, t > 0$ by*

$$p_t^{(\frac{\kappa\delta}{\alpha})}(x,y) = \left(\frac{y}{(ct)^{1/\kappa}}\right)^{\kappa\delta-1} (ct)^{-1/\kappa} \sum_{n=0}^{\infty} \frac{{}_1\Psi_1\left(\begin{array}{c} (1, \frac{n}{\kappa}+\delta) \\ (\kappa, \kappa\delta) \end{array} \middle| -\frac{y^\kappa}{ct}\right)}{n!\Gamma\left(\frac{n}{\kappa}+\delta\right)} \left(-\frac{yx}{(ct)^{2/\kappa}}\right)^n.$$

In particular, the density, with respect to the Lebesgue measure, of the self-similar branching process $(Y, \mathbb{Q}^{(0)})$ is given by

$$p_t^{(0)}(x,y) = \left(\frac{y}{(ct)^{1/\kappa}}\right)^{-1} (ct)^{-1/\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\frac{n}{\kappa}\right)} {}_1\Psi_1\left(\begin{array}{c} (1, \frac{n}{\kappa}) \\ (\kappa, 0) \end{array} \middle| -\frac{y^\kappa}{ct}\right) \left(\frac{yx}{(ct)^{2/\kappa}}\right)^n.$$

For $\delta > 0$, the entrance law of $(Y, \mathbb{Q}^{(\delta)})$ is given by

$$p_t^{(\frac{\kappa\delta}{\alpha})}(0,y) = \frac{t^{-1/\kappa}}{\Gamma(\delta)} \left(\frac{y}{(ct)^{1/\kappa}}\right)^{\kappa\delta-1} {}_1\Psi_1\left(\begin{array}{c} (1, \delta) \\ (\kappa, \kappa\delta) \end{array} \middle| -\frac{y^\kappa}{ct}\right). \tag{4.6}$$

Proof. In what follows, we simply write δ for $\alpha\delta_\kappa$ and set $c = 1$. Thus, by means of the binomial formula, we get, for $t\lambda^\kappa > 1$,

$$(1+t\lambda^\kappa)^{-\delta} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\delta+n)}{n!\Gamma(\delta)} (t\lambda^\kappa)^{-n-\delta}.$$

The term-by term inversion yields

$$p_t^{(\frac{\kappa\delta}{\alpha})}(0,y) = \frac{t^{-1/\kappa}}{\Gamma(\delta)} \left(\frac{y}{t^{1/\kappa}}\right)^{\kappa\delta-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\delta)}{n!\Gamma(\kappa(n+\delta))} \left(\frac{y}{t^{1/\kappa}}\right)^{\kappa n}.$$

Next, we have

$$\begin{aligned} e^{-\lambda x(1+t\lambda^\kappa)^{-\frac{1}{\kappa}}} (1+t\lambda^\kappa)^{-\delta} &= \sum_{n=0}^{\infty} (-1)^n (1+t\lambda^\kappa)^{-(\frac{n}{\kappa}+\delta)} \frac{1}{n!} \lambda^n x^n \\ &= t^{-\delta} \sum_{n=0}^{\infty} (-1)^n (1+(t\lambda^\kappa)^{-1})^{-(\frac{n}{\kappa}+\delta)} \frac{1}{n!} \lambda^{-\kappa\delta} \left(\frac{x}{t^{1/\kappa}}\right)^n. \end{aligned}$$

Once again inverting term and term and using the previous result, we deduce that

$$p_t^{(\frac{\kappa\delta}{\alpha})}(x,y) = t^{-\delta} \sum_{n=0}^{\infty} (-1)^n F_t^n(y) \frac{1}{n!} \left(\frac{x}{t^{1/\kappa}}\right)^n$$

where the term $F_t^n(y)$ is given by

$$\begin{aligned} F_t^n(y) &= \frac{y^{\kappa\delta-1}}{\Gamma\left(\frac{n}{\kappa}+\delta\right)} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r+\frac{n}{\kappa}+\delta)}{r!\Gamma(\kappa(r+\delta))} \left(\frac{y}{t^{1/\kappa}}\right)^{\kappa r} \\ &= \frac{y^{\kappa\delta-1}}{\Gamma\left(\frac{n}{\kappa}+\delta\right)} {}_1\Psi_1\left(\begin{array}{c} (1, \frac{n}{\kappa}+\delta) \\ (\kappa, \kappa\delta) \end{array} \middle| -\frac{y^\kappa}{t}\right). \end{aligned}$$

Note that $F_t^n(y) = G(n) \star p_t^{(\frac{\kappa\delta+n}{\alpha})}(0,y)$. Thus, by putting pieces together we get

$$p_t^{(\frac{\kappa\delta}{\alpha})}(x,y) = \left(\frac{y}{t^{1/\kappa}}\right)^{\delta-1} t^{-1/\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\frac{n}{\kappa}+\delta\right)} {}_1\Psi_1\left(\begin{array}{c} (1, \frac{n}{\kappa}+\delta) \\ (\kappa, \kappa\delta) \end{array} \middle| -\frac{y^\kappa}{t}\right) \left(\frac{yx}{t^{2/\kappa}}\right)^n$$

which completes the proof. $\square$

Note that in the case $\kappa = 1$, we get

$$\begin{aligned} F_t^n(y) &= \frac{y^{\delta-1}}{\Gamma(n+\delta)} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r+n+\delta)}{r!\Gamma(r+\delta)} \left(\frac{y}{t}\right)^r \\ &= \frac{y^{\delta-1}}{\Gamma(n+\delta)} {}_1\Psi_1\left(\begin{array}{c} (1, n+\delta) \\ (1, \delta) \end{array} \middle| -\frac{y^\kappa}{t}\right). \end{aligned}$$

Thus, by means of the formula for products of power series, we recover the well-known expression of the density of the semi-group of a Bessel squared process, see e.g. [6, p.136],

$$\begin{aligned} p_{\frac{t}{2}}^{(\frac{\delta}{2})}(x,y) &= \left(\frac{y}{t}\right)^{\delta-1} t^{-1} e^{-\frac{x+y}{t}} \sum_{n=0}^{\infty} \frac{\left(\frac{xy}{t^2}\right)^n}{n!\Gamma(n+\delta)} \\ &= \left(\frac{y}{xt}\right)^{\frac{\delta-1}{2}} t^{-1} e^{-\frac{x+y}{t}} I_{\delta-1}\left(\frac{2\sqrt{xy}}{t}\right), \end{aligned}$$

where we recall that

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

stands for the modified Bessel function of the first kind of index ν, see e.g. [21].

We proceed by characterizing the Lévy processes associated to $(Y, \mathbb{Q}^{(\delta)})$ via the Lamperti mapping.

Proposition 4.12. *For any $\delta \geq 0$, the Lévy process associated to the κ-sspMp $(Y, \mathbb{Q}^{(\delta)})$ is the Lévy process $(-\xi, \mathbb{P}^{(0,\delta)})$.*

Proof. Let us first consider the case $\delta = 1$. Lambert [17] showed that $(Y, \mathbb{Q}^{(1)})$ corresponds to the branching process $(Y, \mathbb{Q}^{(0)})$ conditioned to never extinct, which is simply the h-transform in the Doob's sense, with $h(x) = x$, of the minimal process $(Y, \mathbb{Q}^{(0)})$. Let us now compute the infinitesimal generator of $(Y, \mathbb{Q}^{(1)})$, denoted by $\mathbf{Q}_+^{(1)}$. To this end, let us recall that since the process X does not have negative jumps and has a finite mean, its infinitesimal generator, denoted by $\mathbf{Q}_+^\dagger$, is given, see also [8], for a smooth function f on $\mathbb{R}^+$ with $f(0) = 0$ and any $x > 0$, by

$$\begin{aligned} \mathbf{Q}_+^\dagger f(x) &= \int_0^\infty \left(f(x+y) - f(x) - yf'(x)\right) \frac{c_-}{y^{\alpha+1}} dy \\ &= x^{-\alpha} \int_1^\infty \left(f(ux) - f(x) - xf'(x)(u-1)\right) \frac{c_-}{(u-1)^{\alpha+1}} du \end{aligned}$$

where we have performed the change of variable $y = x(u-1)$. Thus, by a formula of Volkonskii, see e.g. Rogers and Williams [33, III.21], we deduce that, for a function f as

above and any $x>0$,

$$\begin{aligned}
\mathbf{Q}_+^{(0)} f(x) &= x\mathbf{Q}_+^{\dagger} f(x) \\
&= x\int_0^\infty \left(f(x+y)-f(x)-yf'(x)\right)\frac{c_-}{y^{\alpha+1}}dy \\
&= x^{1-\alpha}\int_1^\infty \left(f(ux)-f(x)-xf'(x)(u-1)\right)\frac{c_-}{(u-1)^{\alpha+1}}du
\end{aligned}$$

Recalling that for any $x>0$, $\mathbf{Q}_+^{(0)}h(x)=0$ with $h(x)=x$ we get, by h-transform and for a smooth function f on $\mathbb{R}^+$, that

$$\begin{aligned}
\mathbf{Q}_+^{(1)} f(x) &= \frac{1}{h(x)}\mathbf{Q}_+^{(0)}(hf)(x) \\
&= x^{1-\alpha}\left(\mathbf{Q}_+^{(0)} f(x)+\int_0^\infty \left(f(x+y)-f(x)\right)\frac{c_-}{y^{\alpha}}\right)dy \\
&= x^{1-\alpha}\left(\mathbf{Q}_+^{(0)} f(x)+\int_1^\infty \left(f(ux)-f(x)\right)\frac{c_-}{(u-1)^{\alpha}}du\right) \\
&= x^{1-\alpha}\left(\mathbf{Q}_+^{(0)} f(x)+\int_0^1 \left(f\left(\frac{x}{u}\right)-f(x)\right)\frac{c_-u^{\alpha-2}}{(1-u)^{\alpha}}du\right)
\end{aligned}$$

We have already shown, see Lemma 4.3, that the Lévy process associated via the Lamperti mapping to $(Y,\mathbb{Q}_+^{(0)})$ is $(-\xi,\mathbb{P}^{(0)})$. Next, consider the function $p_\lambda(x)=x^\lambda$, with $\lambda<0$ and $x>0$, and note that $\mathbf{Q}_+^1 p_\lambda(x)=x^{\lambda-\alpha}\psi(-\lambda)$. Thus, using the integral (2.4) we obtain that

$$\begin{aligned}
\int_0^1 \left(u^{-\lambda}-1\right)\frac{c_\alpha u^{\alpha-2}}{(1-u)^{\alpha}}du &= (-\lambda)_\alpha\frac{c_\alpha\Gamma(1-\alpha)}{-\lambda+\kappa} \\
&= -c(-\lambda)_\alpha\frac{1}{-\lambda+\kappa}
\end{aligned}$$

Using the recurrence formula of the Gamma function, we deduce that the image, via the Lamperti mapping, of $(Y,\mathbb{Q}^{(1)})$ is $(-\xi,\mathbb{P}^{(0,1)})$. The general case is deduced from the previous one by recalling that for any $\delta,x,\lambda>0$, and writing $e_\lambda(x)=e^{-\lambda x}$, $\lambda\geq 0$, we have, see e.g. [38], for any $x>0$,

$$\mathbf{Q}_+^{(\delta)}e_\lambda(x) = -e_\lambda(x)\left(x\varphi(\lambda)+\delta\chi(\lambda)\right)$$

where $\mathbf{Q}_+^{(\delta)}$ stands for the infinitesimal generator of $(Y,\mathbb{Q}^{(\delta)})$. $\square$

Remark 4.13. We observe that the process $(-\xi,\mathbb{P}^{(0,1)})$ is equivalent to $(-\xi,\mathbb{P}^{(-1)})$. This is not really surprising since as mentioned in the proof, the process $(Y,\mathbb{Q}^{(1)})$ is obtained from $(Y,\mathbb{Q}^{(0)})$ by h-transform with $h(x)=x$. The corresponding Lévy process is thus the $\theta=1$-Esscher transform of $(-\xi,\mathbb{P}^{(0)})$ which is $(-\xi,\mathbb{P}^{(-1)})$.

The Theorem is proved by putting pieces together and using the following easy result.

Lemma 4.14. *For any $v>0$ and $\delta>\frac{\kappa}{\alpha}$, we have*

$$f_\infty^{(\delta)}(v)=|\mathbb{E}^{(0,\delta)}[\xi_1]|v^{-\frac{1}{\kappa}}p_1^{(\delta)}(0,v^{-\frac{1}{\kappa}}).$$

Proof. In [29, Lemma 3.2], the following identity is proved

$$\mathbb{E}^{(0,\delta)}\left[e^{-qy^{\kappa}\Sigma_{\infty}}\right] = |\mathbb{E}^{(0,\delta)}[\xi_1]| \int_0^{\infty} e^{-qt} p_1^{(\delta)}(0, yt^{-\frac{1}{\kappa}}) y^{1-\kappa} dy.$$

Performing the change of variable $t = y^{\alpha} v$, the proof is completed by invoking the injectivity of the Laplace transform. $\square$

Recalling that for $\delta > \frac{\kappa}{\alpha}$ we have $\psi(\theta) = 0$ where $\theta = \alpha\delta - \kappa$. We deduce readily, from Rivero [32], the behavior of $(Y, \mathbb{Q}^{(\delta)})$ at the boundary point 0.

Proposition 4.15. *(1) For $\delta \geq \frac{\kappa}{\alpha}$, 0 is unattainable.*

(2) For $\delta < \frac{\kappa}{\alpha}$, 0 is reached a.s.. Moreover, if $0 < \delta < \frac{\kappa}{\alpha}$, the boundary 0 is recurrent and reflecting, i.e. there exists an unique recurrent extension of the minimal process which hits and leaves 0 continuously a.s. and which is κ-self-similar on $[0, \infty)$.

(3) For $\delta = 0$, the point 0 is a trap.

Remark 4.16. Note that in the diffusion case, i.e. $\kappa = 1$, there is an absolute continuity relationship between the family of law $(\mathbb{Q}^{(\delta)})_{\delta>0}$. More precisely, we have for any $\delta, \delta_1 > 0$

$$d\mathbb{Q}_x^{(\delta_1)}{}_{|F_t} = e^{-\delta(Y_t - x) - \frac{\delta^2}{2}\int_0^t Y_s^{-1} ds} d\mathbb{Q}_x^{(\delta_1+\delta)}{}_{|F_t}, \quad x, t > 0. \tag{4.7}$$

This follows readily by a time change (via the Lamperti mapping) of the Cameron-Martin formula which relates the laws of a Brownian motion with different drifts. In the general case, the Esscher transform studied earlier does not yield such an absolute continuity relationship since the family of laws of Lévy processes $(\xi, \mathbb{P}^{(0,\delta)})_{\delta\geq 0}$ are not related by such a relationship.

We proceed by characterizing the class of sspMp which enjoy the infinite decomposability property introduced by Shiga and Watanabe [36]. More specifically, let $\mathfrak{D}_M^+$ the Skorohod space of nonnegative valued homogeneous Markov process with càdlàg paths. Let $(\mathbb{Q}_x)_{x\geq 0}, (\mathbb{Q}_x^1)_{x\geq 0}$ and $(\mathbb{Q}_x^2)_{x>0}$ be three systems of probability measures defined on $(\mathfrak{D}_M^+, \mathfrak{B}(\mathfrak{D}_M^+))$, where $\mathfrak{B}(\mathfrak{D}_M^+)$ be the σ-field on $\mathfrak{D}_M^+$ generated by the Borel cylinder sets. Then, define

$$\mathbb{Q} = \mathbb{Q}^1 * \mathbb{Q}^2 \tag{4.8}$$

if and only if for every $x, y \geq 0$, $\mathbb{Q}_{x+y} = \Phi(\mathbb{Q}_x^1 \times \mathbb{Q}_x^2)$ where Φ is the mapping $\mathfrak{D}_M^+ \times \mathfrak{D}_M^+ \to \mathfrak{D}_M^+$ defined by

$$\Phi(x_1, x_2) = x_1 + x_2. \tag{4.9}$$

A positive valued Markov process with law $\mathbb{Q}$ is infinitely decomposable if for every $n \geq 1$ there exits a $\mathbb{Q}^n$ such that

$$\mathbb{Q} = \underbrace{\mathbb{Q}^n * \ldots * \mathbb{Q}^n}_{n}. \tag{4.10}$$

They showed that there is one to one mapping between cbip and conservative Markov processes having the property (4.10). Thus, we get the following result.

Corollary 4.17. *There is one to one mapping between the family of sspMp satisfying the Shiga-Watanabe infinite decomposability property* (4.10) *and the family* $(Y, \mathbb{Q}^{(\delta)})_{\delta>0}$*, i.e. the family of spectrally positive sspMp associated, via the Lamperti mapping, to the family of Lévy processes* $(-\xi, \mathbb{P}^{(0,\delta)})_{\delta>0}$.

Finally, we provide an extension of the previous result the κ-sspMp Ornstein-Uhlenbeck processes. More specifically, for any $\eta \in \mathbb{R}$, let us introduce the family of laws $(\mathbb{Q}^{\eta,(\delta)})_{\delta\geq 0}$ of self-similar Ornstein-Uhlenbeck processes associated to $(Y, \mathbb{Q}^{(\delta)})_{\delta\geq 0}$ by the following time-space transform, for any $t \geq 0$ and $\delta \geq 0$,

$$\left(Y_t, \mathbb{Q}^{\eta,(\delta)}\right) = \left(e^{-\eta t} Y_{\tau_{-\eta}(t)}, \mathbb{Q}^{(\delta)}\right),$$

where

$$\tau_\eta(t) = \frac{1 - e^{-\eta\kappa t}}{\eta\kappa}.$$

For any $\eta \in \mathbb{R}$, $(Y, \mathbb{Q}^{\eta,(\delta)})$ is an homogenous Markov process and for $\eta > 0$, it has an unique stationary measure which is the entrance law of $(Y, \mathbb{Q}^{(\delta)})$, see e.g. [30]. Moreover, its semi-group is characterized by its Laplace transform as follows

$$\mathbf{E}_x\left[e^{-\lambda U_t}\right] = (1 + c\tau_\eta(t)\lambda^\kappa)^{-\delta/\kappa} \, e^{-xe^{-\eta t}\lambda(1+c\tau_\eta(t)\lambda^\kappa)^{-1/\kappa}},$$

It is easily shown that the infinitesimal generator of $(Y, \mathbb{Q}^{\eta,(\delta)})$ has the following form

$$\mathbf{Q}_+^{\eta,(\delta)} f(x) = \mathbf{Q}_+^{(\delta)} f(x) - \eta x f'(x)$$

for a smooth function f. Hence we deduce from the identity

$$\mathbf{Q}_+^{(\delta)} e_\lambda(x) \quad = \quad -e_\lambda(x) \left(x\varphi(\lambda) + \delta\chi(\lambda)\right),$$

that $(Y, \mathbb{Q}^{\eta,(\delta)})$ is a cbip with branching mechanism $\varphi_\eta(\lambda) = \varphi\lambda - \eta\lambda$ and the immigration mechanism $\chi(\lambda)$. Its semi-group is absolutely continuous with a density denoted $p_t^{(\delta,\eta)}(x,y)$ and given, for any $x, y, t > 0$, by

$$p_t^{(\delta,\eta)}(x,y) \quad = \quad e^{-\kappa\eta t} p^{(\delta)}_{\tau_{-\eta}(t)}(x, e^{-\kappa\eta t} y).$$

5. **Some concluding remarks**

5.1. **Representations of some ${}_p\Psi_q$ functions.** Let us recall that, for $\frac{\kappa}{\alpha} < \delta < \frac{2\alpha-2}{\alpha}$, the Laplace transform of $(\Sigma_\infty, \mathbb{P}^{(0,\delta)})$ has been computed by Patie [29] as follows, for any $x \geq 0$,

$$\mathbb{E}^{(0,\delta)}\left[e^{-x\Sigma_\infty}\right] \quad = \quad \mathcal{N}_{\kappa,\delta}(x). \tag{5.1}$$

where, by setting $0 < m_\kappa = 2 - \frac{\alpha\delta}{\kappa} < 1$,

$$\mathcal{N}_{\kappa,\delta}(x) = \mathcal{I}_{\kappa,\delta}\left(x\right) - C_{m_\kappa} x^{\frac{\alpha\delta}{\kappa}-1} \mathcal{I}_{\kappa,\delta,m_\kappa}\left(x\right), \quad x \geq 0,$$

and

$$
\begin{aligned}
\mathcal{I}_{\kappa,\delta}\left(ckx\right) &= \Gamma(m_\kappa)\Gamma(\kappa)\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+m_\kappa)\Gamma(\kappa(n+1))} \\
&= \Gamma(m_\kappa)\Gamma(\kappa)\,{}_1\Psi_2\left(\begin{array}{c}(1,1)\\(1,m_\kappa)(\kappa,\kappa)\end{array}\middle|\, x\right), \\
\mathcal{I}_{\kappa,\delta,\theta}\left(ckx\right) &= \Gamma(\alpha\delta)\sum_{n=0}^{\infty}\frac{x^n}{n!\Gamma(\kappa n+\alpha\delta)} \\
&= \Gamma(\alpha\delta)\quad {}_0\Psi_1\left(\ (\kappa,\alpha\delta)\ \middle|\ x\right),
\end{aligned}
$$

and where C_{m_κ} is determined by

$$
\mathcal{I}_{\kappa,\delta}\left(x\right) \sim C_{m_\kappa} x^{\frac{\alpha\delta}{\kappa}-1}\mathcal{I}_{\kappa,\delta,m_\kappa}\left(x\right) \quad \text{as } x\to\infty. \tag{5.2}
$$

Using the exponentially infinite asymptotic expansions (6.2) and (6.3), we deduce that

$$
C_{m_\kappa} = \frac{\Gamma(m_\kappa)\Gamma(\kappa)}{\Gamma(\alpha\delta)}.
$$

Observing that $\lim_{\lambda\to\infty}\frac{\psi(\lambda)}{\lambda^{\kappa+1}} = c$, we also have, see [29],

$$
C_{m_\kappa} = \frac{\Gamma(m_\kappa)}{\kappa}c^{(m_\kappa-1)}e^{E_\gamma\kappa(m_\kappa-1)}\prod_{k=1}^{\infty}e^{-\frac{\kappa+\alpha\delta}{k}}\frac{(k+1-m_\kappa)\psi^{(0,\delta)}(\kappa k)}{k\psi^{(0,\delta)}(\kappa k+1-m_\kappa)}
$$

where $E_\gamma = 0.577\ldots$ stands for Euler-Mascheroni constant.

As a consequence of the Theorem 4.7 we have this interesting representation of the function $\mathcal{N}_{\kappa,\delta}(x)$ under consideration.

Corollary 5.1. *For any $\delta > \frac{\kappa}{\alpha}$, the density $f_\infty^{(\delta)}$ is a mixture of exponential distribution, in particular is log-convex. Moreover, for $\frac{\kappa}{\alpha} < \delta < \frac{2\alpha-2}{\alpha}$, we have the following two representations,*

$$
\begin{aligned}
\mathcal{N}_{\kappa,\delta}(x) &= \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)}\int_0^\infty \frac{u^{\alpha\delta_\kappa-1}}{x+u}\quad {}_0\Psi_1\left(\ (\kappa,\alpha\delta)\ \middle|\ -u\right)du. \\
\mathcal{N}_{\kappa,\delta}(x) &= \exp\left(-\int_0^\infty(1-e^{-ux})L(u)du\right)
\end{aligned}
$$

where $L(u) = \int_0^\infty e^{-ur}q(r)dr$ with $0\le q(r)\le 1$ a measurable function and $\int_0^1\frac{q(r)}{r}dr<\infty$. Finally, we have the following asymptotic expansion for large x,

$$
\mathcal{N}_{\kappa,\delta}(x) \sim \exp\left(-(\kappa^{-\kappa}\alpha^\alpha x)^{\frac{1}{\alpha}}\right)x^{-\frac{\delta}{\kappa}-\frac{1}{\alpha}+1}.
$$

Proof. Note that, for any $\delta > \frac{\kappa}{\alpha}$ and $y>0$, we have

$$
\begin{aligned}
f_\infty^{(\delta)}(y) &= \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)}\int_0^\infty e^{-uy}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\kappa n+\alpha\delta)}u^{n+\alpha\delta_\kappa-1}du \\
&= \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)}\int_0^\infty e^{-uy}u^{\alpha\delta_\kappa-1}\quad {}_0\Psi_1\left(\ (\kappa,\alpha\delta)\ \middle|\ -u\right)du.
\end{aligned}
$$

Thus $f_\infty^{(\delta)}$ is a completely monotone function. The fact that $f_\infty^{(\delta)}$ is a mixture of exponential distribution follows from Proposition 51.8 in Sato [35]. The first representation is obtained by taking the Laplace transform on both sides on the previous equation and using (5.1). The last one follows from [35, Theorem 51.12]. The asymptotic behavior of the function $\mathcal{N}_{\kappa,\delta}$ is deduced from the exponentially infinite asymptotic expansions (6.2) and (6.3). $\square$

Next, since the law of $\left(\Sigma_\infty, \mathbb{P}^{(0,\delta)}\right)$ is self-decomposable, see e.g. [29], it is unimodal. In other words, the derivative of the continuously differentiable density admits a unique zero. We state this fact in the following.

Corollary 5.2. *For any $\delta > \frac{\kappa}{\alpha}$ with $0 < \kappa \le 1$, the function*

$$x \mapsto {}_1\Psi_1\left(\begin{array}{c} (1, 1+\alpha\delta_\kappa) \\ (\kappa, \alpha\delta) \end{array} \middle| -x \right)$$

admits a unique zero on $\mathbb{R}^+$.

Remark 5.3. In the case $\kappa = 1$, the result above is obvious since

$${}_1\Psi_1\left(\begin{array}{c} (1, 1+\alpha\delta) \\ (1, \alpha\delta) \end{array} \middle| -x \right) = e^{-x}\left(\alpha\delta - x\right).$$

6. **Asymptotic expansions of the Wright Hypergeometric functions**

Special classes of the Wright generalized hypergeometric functions have been considered among others by Mittag-Leffler [26], Barnes [1], Fox [13] while the general case ${}_p\Psi_q$ has been considered by Wright [37]. We refer to Braaksma [7, Chap. 12] for a detailed account of this function and its relation to the G-function. In the sequel, we simply indicate special properties, which can be found in [7, Chap. 12].

We proceed by recalling that the Wright hypergeometric function is defined as

$${}_p\Psi_q\left(\begin{array}{c} (A_1, a_1), \dots (A_p, a_p) \\ (B_1, b_1) \dots (B_q, b_q) \end{array} \middle| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(A_i n + a_i)}{\prod_{i=1}^{q} \Gamma(B_i n + b_i)} \frac{z^n}{n!}$$

where p, q are nonnegative integers, $a_i \in \mathbb{C}\,(i = 1 \dots p), b_j \in \mathbb{C}\,(j = 1 \dots q)$, the coefficients $A_i \in \mathbb{R}^+$ $(i = 1 \dots p)$ and $B_j \in \mathbb{R}^+$ $(j = 1 \dots q)$ are such that

$$A_i n + a_i \neq 0, -1, -2, \dots \qquad (i = 0, 1, \dots, p; n = 0, 1, \dots).$$

In what follows, we will also use the number S and T defined respectively by

$$\begin{aligned} S &= 1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \\ T &= \prod_{i=1}^{p} A_i^{A_i} \prod_{i=1}^{q} B_i^{-B_i}. \end{aligned}$$

Throughout this part we assume that S is positive. In such a case, the series is convergent for all values of z and it defines an integral function of z (the case $S = 0$ is also treated in [7]). Next, for S positive, the function ${}_p\Psi_q$ admits a contour integral representation. More precisely, we have

$$ {}_p\Psi_q\left(\begin{array}{c} (A_1,a_1),\ldots(A_p,a_p) \\ (B_1,b_1)\ldots(B_q,b_q) \end{array} \middle| \, z \right) = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\prod_{i=1}^{p}\Gamma(A_i n + a_i)}{\prod_{i=1}^{q}\Gamma(B_i n + b_i)}\Gamma(-s)(-s)^z ds $$

where $\mathfrak{C}$ is a contour in the complex s-plane which runs from $s = a - i\infty$ to $s = a + i\infty$ (a an arbitrary real number) so that the points $s = 0, 1, 2\ldots$ and $s = -\frac{a_j+n}{A_i}$, $(i = 0, 1, \ldots, p; n = 0, 1, \ldots)$ lie to the right left of $\mathfrak{C}$. Next we introduce the following functions

$$ \begin{aligned} P(z) &= \sum_{s\in R_p} z^s \Gamma(-s) Res\left(\frac{\prod_{i=1}^{p}\Gamma(A_i s + a_i)}{\prod_{i=1}^{q}\Gamma(B_i s + b_i)}\right) \\ E(z) &= \frac{\exp\left((TS^S z)^{\frac{1}{S}}\right)}{S}\sum_{k=0}^{\infty} H_k (TS^S z)^{\frac{1-G-k}{S}} \end{aligned} $$

where Res stands for residuum, we set $R_p = \{r_{i,n} = -\frac{a_i+n}{A_i}, i = 0, 1, \ldots, p; n = 0, 1, \ldots\}$ and the constant G is given by

$$ G = \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i + \frac{p-q}{2} + 1. $$

The coefficients $(H_k)_{k\geq 0}$ are determined by

$$ \frac{\prod_{i=1}^{p}\Gamma(A_i s + a_i)}{\prod_{i=1}^{q}\Gamma(B_i s + b_i)}(TS^S)^{-s} \sim \sum_{k=0}^{\infty}\frac{H_k}{\Gamma(k + Ss + G)}. $$

In particular,

$$ H_0 = (2\pi)^{\frac{p-q}{2}} S^{G-\frac{1}{2}} \prod_{i=1}^{p} A_i^{a_i - \frac{1}{2}} \prod_{i=1}^{q} B_i^{\frac{1}{2} - b_i}. $$

We have the following asymptotic expansions.

(1) Suppose $S > 0$ and $p > 0$. Then, the following algebraic asymptotic expansion

$$ {}_p\Psi_q(z) \sim P(-z) $$

holds for $\mid z \mid \to \infty$ uniformly on every closed subsector of

$$ \mid \arg(-z) \mid < \left(1 - \frac{S}{2}\right)\pi $$

(2) Suppose $S > 0$. Then, the following exponentially infinite asymptotic expansion

$$ {}_p\Psi_q(z) \sim E(z) $$

holds for $\mid z \mid \to \infty$ uniformly on every closed sector (vertex in 0) contained in $\arg(z) < \min(S, 2)\frac{\pi}{2}$.

For the convenience of the reader, we list below the asymptotic expansion corresponding to ${}_p\Psi_q$ functions which appear in this paper. For $y \to \infty$, we have

$$(6.1) \qquad {}_1\Psi_1\left(\begin{array}{c} (1, \alpha\delta_\kappa) \\ (\kappa, \alpha\delta) \end{array} \middle| -y^\kappa \right) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha\delta_\kappa)}{\Gamma(-\kappa n)} \frac{y^{-\alpha\delta_\kappa - n}}{n!},$$

$$(6.2) \qquad {}_1\Psi_2\left(\begin{array}{c} (1,1) \\ (1, m_\kappa)(\kappa,\kappa) \end{array} \middle| y \right) \sim (2\pi\alpha)^{-\frac{1}{2}} \kappa^{\frac{1}{2}-\delta-\frac{\kappa}{2\alpha}} \exp\left((\kappa^{-\kappa}\alpha^\alpha y)^{\frac{1}{\alpha}} \right) y^{\frac{\delta}{\kappa}+\frac{1}{\alpha}-1}$$

$$(6.3) \qquad {}_0\Psi_1\left(\, (\kappa, \alpha\delta) \, \middle| \, y \right) \sim (2\pi\alpha)^{-\frac{1}{2}} \kappa^{\frac{1}{2}-\delta-\frac{\kappa}{2\alpha}} \exp\left((\kappa^{-\kappa}\alpha^\alpha y)^{\frac{1}{\alpha}} \right) y^{\frac{1}{2\alpha}-\delta}.$$

References

[1] E.W. Barnes. The asymptotic expansion of integral functions defined by generalized hypergeometric series. *Proc. London Math. Soc.*, 5(2):59–116, 1907.

[2] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, 1996.

[3] J. Bertoin, P. Biane, and M. Yor. Poissonian exponential functionals, q-series, q-integrals, and the moment problem for log-normal distributions. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, volume 58 of *Progr. Probab.*, pages 45–56. Birkhäuser, Basel, 2004.

[4] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002.

[5] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005.

[6] A.N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, 2^{nd} edition, 2002.

[7] B.L.J. Braaksma. Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compositio Math.*, 15:239–341, 1964.

[8] M.E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006.

[9] Ph. Carmona, F. Petit, and M. Yor. Exponential functionals of Lévy processes. *Lévy processes Theory and Applications*, pages 41–55, 2001.

[10] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Ann. Appl. Probab.*, 13(3):984–1053, 2003.

[11] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990.

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Higher Transcendental Functions*, volume 3. McGraw-Hill, New York-Toronto-London, 1955.

[13] C. Fox. The asymptotic expansion of generalized hypergeometric functions. *Proc. London Math. Soc.*, 27(2):389–400, 1928.

[14] R.K. Getoor and M.J. Sharpe. Naturality, standardness, and weak duality for Markov processes. *Z. Wahrsch. verw. Gebiete*, 67:1–62, 1984.

[15] I.S. Gradshteyn and I.M. Ryshik. *Table of Integrals, Series and Products*. Academic Press, San Diego, 6^{th} edition, 2000.

[16] D.R. Grey. Asymptotic behaviour of continuous time, continuous state-space branching processes. *J. Appl. Probability*, 11:669–677, 1974.

[17] A. Lambert. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.*, 12:420–446, 2007.

[18] J. Lamperti. Continuous state branching processes. *Bull. Amer. Math. Soc.*, 73:382–386, 1967.

[19] J. Lamperti. Limiting distributions for branching processes. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part*, pages 225–241. Univ. California Press, Berkeley, Calif., 1967.

[20] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Geb.*, 22:205–225, 1972.

[21] N.N. Lebedev. *Special Functions and their Applications*. Dover Publications, New York, 1972.

[22] Ju. V. Linnik. Linear forms and statistical criteria. I. In *Selected Transl. Math. Statist. and Prob., Vol.*, pages 1–40. Amer. Math. Soc., Providence, R.I., 1963.
[23] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, 2:312–347, 2005.
[24] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. II. Some related diffusion processes. *Probab. Surv.*, 2:348–384, 2005.
[25] M.S. Milgram. On Hypergeometric 3F2(1). *available as http://www.arXiv.org:math:CA/0603096.*
[26] G. Mittag-Leffler. Sur la nouvelle function $E_\alpha(x)$. *C. R. Math. Acad. Sci. Paris*, 137:554–558, 1903.
[27] Y.A. Neretin. Stable densities and operators of fractional differentiation. In *Representation Theory, Dynamical Systems, and Asymptotic Combinatorics*, volume 217 of *Amer. Math. Soc. Transl. Ser.*, pages 117–137. Amer. Math. Soc., Providence, RI, 2006.
[28] A.G. Pakes. Mixture representations for symmetric generalized Linnik laws. *Statist. Probab. Lett.*, 37(3):213–221, 1998.
[29] P. Patie. Infinitely divisibility of solutions to some semi-stable integro-differential equations and exponential functionals of Lévy processes. *Submitted*, 2007.
[30] P. Patie. q-invariant functions associated to some generalizations of the Ornstein-Uhlenbeck semigroup. *Under revision for ALEA*, 2007.
[31] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293. Springer-Verlag, Berlin-Heidelberg, 3^{rd} edition, 1999.
[32] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005.
[33] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Vol. 1.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Foundations, Reprint of the second (1994) edition.
[34] G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes*. Stochastic Modeling. Chapman & Hall, New York, 1994.
[35] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.
[36] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 27:37–46, 1973.
[37] E.M. Wright. The asymptotic expansion of the generalized hypergeometric function. *Proc. London Math. Soc. (2)*, 46:389–408, 1940.
[38] L. Zeng-Hu. Branching processes with immigration and related topics. *Front. Math. China*, 1(1):73–97, 2006.

Department of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse, 5, CH-3012 Bern, Switzerland
E-mail address: patie@stat.unibe.ch